Introduction to Quantum Computing

Tutorial 6 - Solutions

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Problem 1

Recall from the lectures the definition of the $J$ gate as:

$$ J(\theta) = HR(\theta) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & e^{i\theta} \\ 1 & -e^{i\theta} \end{pmatrix} $$

Additionally, recall that any single qubit gate can be decomposed using $J$ gates: $U = J(0)J(\theta_1)J(\theta_2)J(\theta_3)$, for some $\theta_1, \theta_2, \theta_3$.

A) Find the $J$ decompositions (i.e. find the angles $\theta_1, \theta_2, \theta_3$) for the gates $Z$ and $X$.

**Solution:** In certain cases we can use various heuristic techniques to find the theta parameters. These would entail using things like $J(0) = H$, $HH = I$, $HXH = Z$, $HZH = X$ etc. However, we will calculate the general expression for $U$ from the decomposition and equate it to $Z$ and $X$ respectively, to determine the values of the theta parameters. As mentioned, we have that $J(0) = H$ and that $HH = I$, and so our expression for $U$ becomes:

$$ U = HHR(\theta_1)J(\theta_2)J(\theta_3) = R(\theta_1)J(\theta_2)J(\theta_3) $$

If we do the explicit matrix multiplication, we find that:

$$ U = \frac{1}{2} \begin{pmatrix} 1 + e^{i\theta_2} & e^{i\theta_3} - e^{i(\theta_1 + \theta_3)} \\ e^{i\theta_1} - e^{i(\theta_1 + \theta_2)} & e^{i(\theta_1 + \theta_2 + \theta_3)} \end{pmatrix} $$

It is now a simple matter of identifying the matrix elements corresponding to the $Z$ and $X$ operators and picking the appropriate values for $\theta_1, \theta_2, \theta_3$. It is easy to notice that for $Z$ we have $\theta_1 = \pi, \theta_2 = \theta_3 = 0$ and for $X$ we have $\theta_1 = \theta_3 = 0$ and $\theta_2 = \pi$.

B) Using these decompositions draw the measurement patterns implementing the gates $Z$ and $X$.

**Solution:** Since we know from the lectures how to equate a composition of $J$ gates with an MBQC pattern, we have for $Z$:
Problem 2

Given the following pattern in which the input is \( |\psi\rangle = \frac{1}{\sqrt{2}} (|1\rangle - |0\rangle) \):

Find the corresponding quantum gate of this pattern and the output state.

**Solution:** Similar to the previous exercise, we know from the lecture that the given measurement pattern is a composition of \( J \) gates acting on the first qubit, with the \( \theta \) values given by the measurement angles. In this case, after performing the necessary corrections, the output state will be:

\[
|\phi\rangle = J(-\pi/2)J(\pi)J(\pi/2)J(0) |\psi\rangle
\]

Note that is in the reverse order of the ordering of the measurement angles. It is easy to see why, performing an \( X \) measurement on the first qubit means the second qubit becomes (after correction) \( J(0) |\psi\rangle \). When we then perform a \(-\pi/2\) measurement on this qubit, after correction the third one becomes \( J(\pi/2)J(0) |\psi\rangle \). And so on, until we reach the last qubit. Let us now compute this product of unitaries acting on \( |\psi\rangle \).

\[
U = J(-\pi/2)J(\pi)J(\pi/2)J(0) = \frac{1}{4} \left( \begin{array}{cc} 1 & -i \\ i & 1 \end{array} \right) \left( \begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & i \\ 1 & -i \end{array} \right) \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right)
\]

Performing the computation yields:

\[
U = \frac{1}{4} \begin{pmatrix} 0 & -4i \\ 4i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
\]
Which we can identify as the Pauli $Y$ gate. Computing the action of this on $|\psi\rangle = \frac{i}{\sqrt{2}}(|1\rangle - |0\rangle)$ yields:

$$Y |\psi\rangle = Y \frac{i}{\sqrt{2}}(|1\rangle - |0\rangle) = \frac{(-i)i}{\sqrt{2}} |0\rangle - \frac{ii}{\sqrt{2}} |1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = +$$

**Problem 3**

Give a measurement pattern that takes 2 qubits as input and produces as output the familiar Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

**Solution:** Recall that when we have two $|+\rangle$ states and we entangle them with a $cZ$ operation, we end up with the state:

$$\frac{|+0\rangle + |-1\rangle}{\sqrt{2}}$$

This is almost the $|\Phi^+\rangle$ and in fact can be made into $|\Phi^+\rangle$ by simply applying a Hadamard to either qubit. Our idea for a measurement pattern is then to simply entangle the two $|+\rangle$ input qubits and then apply a Hadamard operation to the second. The graph would then look like this:

Essentially, the procedure for obtaining this Bell state is the following:

- We start with input qubits $|+\rangle_1$ and $|+\rangle_2$
- Add additional qubit $|+\rangle_3$
- Entangle qubits 1 2 and 2 3 with $cZ$ operations
- Measure qubit 2 in $X$ basis
- Apply correction to qubit 3 if necessary
- Bell state is given by qubits 1 and 3