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Need a unique graph state for all computations (of given size)

- How to prepare the input
- How to break a large graph state to smaller: The role of the $Z$-measurement

**How to prepare the input.**
- Assume a *known* input $|\psi\rangle$.
- Use MBQC to prepare $|\psi\rangle$ from $|+\rangle$
- Continue with MBQC pattern as before
  i.e. find $U$ such that $U|+\rangle = |\psi\rangle$, then implement $U$ using MBQC and continue as with if we had the input $|\psi\rangle$

Need to take into account (random) corrections (see later)
The role of the $Z$-measurement.
- We can disconnect extra qubits from the graph in a correctable way
- Measuring a qubit in the $Z := \{|0\rangle, |1\rangle\}$ basis results:
  1. Disconnecting the qubit from the graph (break the edges)
  2. Add a $Z$-correction to neighbours if outcome $s_i = 1$

A $Z$-correction at qubit $j$ does NOT change the measurement required. It only changes $|+\theta\rangle \rightarrow |-\theta\rangle$ and visa-versa. Can take into account by flipping the value of $s_j$

We can perform first all the $Z$ measurements (reducing the resource to the graph needed), and then keep track of the outcome corrections in the remaining pattern.
Example:

- Consider first the $Z$ measurement at qubit $A$
- Need to account for operations that involve qubit $A$, before performing the measurement:

$$\Lambda Z_A |+\rangle_A |+\rangle_2 = |0\rangle_A |+\rangle_2 + |1\rangle_A |-\rangle_2$$
Towards a Universal Resource

Measuring $A$ at the $Z$ basis result:

$$|s_A\rangle Z^{s_A} |+\rangle$$

since $s_A = 1$ leads to $|{-}\rangle_2 = Z |{+}\rangle_2$.

**Net result:** this measurement **removed the vertex $A$ from the graph**, breaking the corresponding edge(s), while resulted to a $Z$ correction to the neighbouring qubit (in the case $s_A = 1$).

**Summary:**
- Measuring qubit $i$ in $|\pm \theta\rangle$ basis results to a gate $J(–\theta)$ and a correction $X^{s_i}$ at the “next” qubit. This correction affects the new measurement angle (in order to cancel it).
- Measuring qubit $i$ in $\{|0\rangle, |1\rangle\}$ basis results to no gate being applied. Only a correction $Z^{s_i}$ being applied to all neighbours. This correction changes the labelling of the outcome, but not the measurement (can be “cancelled” easily).
Corrections appear when \( s_i \neq 0 \)

Let \( \phi_i \) be the measurement angles that implement the desired unitary if all measurements give the result zero \( s_i = 0 \ \forall \ i \)

The set \( \{\phi_i\}_i \) determine the computation performed, and we call them default measurement angles.

We define \( \phi'_i(\phi_i, s_j | j \in \{\text{past of } i\}) \) to be the corrected measurement angles (see later for expression)

Default angles determine computation, corrected angles are the one used for measurements
Graph state $|G\rangle$ is defined as:

$$|G\rangle = \prod_{(a,b) \in E} \wedge Z^{(a,b)} |+\rangle \otimes V$$

An operator $A$ stabilises a state if

$$A |\psi\rangle = |\psi\rangle$$

The state $|G\rangle$ is a **stabiliser state** with generators:

$$K_i := X_i \left( \prod_{j \in N_G(i)} Z_j \right)$$

For each vertex $i \in V$ there is a stabiliser that has $X$ at that vertex and $Z$ to all its neighbours $N_G(i)$ in the graph.
Graph States as Stabiliser States

Operators $K_i$ stabilise $|G\rangle$:

$$K_i |G\rangle = X_i \left( \prod_{j \in N_G(i)} Z_j \right) \prod_{(a,b) \in E} \wedge Z^{(a,b)} |+\rangle \otimes V$$

$$= X_i \prod_{(a,b) \in E} \wedge Z^{(a,b)} |+\rangle \otimes V \setminus N_G(i) |-\rangle \otimes j \in N_G(i)$$

where we used that $Z$ commutes with $\wedge Z$ and that $Z |+\rangle = |-\rangle$. 
Graph States as Stabiliser States

We know that $X_i \land Z^{(i,j)} = \land Z^{(i,j)} X_i Z_j$ (see previous lecture) and we get:

$$K_i |G\rangle = \prod_{(a,b) \in E} \land Z^{(a,b)} \left( X_i \prod_{j \in N_G(i)} Z_j \right) |+\rangle \otimes V \setminus N_G(i) |-\rangle \otimes j \in N_G(i)$$

since $X_i$ acts as above if $i$ belongs to that edge while it commutes with all the other $\land Z$ that do not involve qubit $i$. However this changes back the states since $Z |-\rangle = |+\rangle$, and $X |+\rangle = |+\rangle$ results to

$$K_i |G\rangle = \prod_{(a,b) \in E} \land Z^{(a,b)} |+\rangle \otimes V, \; \forall \; i \in V = |G\rangle$$

- It can be shown that this set of generators uniquely determines the graph state $|G\rangle$. 

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Lecture 15 & 16: MBQC II & III
Flow

Required for an order that measurements and corrections take place

**Definition:** An entanglement graph \((G, I, O)\) has flow if there exists a map \(f : O^c \rightarrow I^c\) and a partial order \(\preceq\) over qubits

1. \(x \sim f(x)\): (\(x\) and \(f(x)\) are neighbours in the graph)
2. \(x \preceq f(x)\): (\(f(x)\) is to the future of \(x\) with respect to the partial order)
3. for all \(y \sim f(x)\), we have \(x \preceq y\): (any other neighbours of \(f(x)\) are all to the future of \(x\))

- Measurements should respect the partial order.
- Condition 3 guarantees no loops: by measuring \(x\) before \(f(x)\) we will never have some \(y\) that \(f(f(x)) = y\) and \(y \preceq x\).

**Example:** 2-dim lattice. \(f(x) \Rightarrow\) same row, next column
How to apply an operator by acting on different qubits

Given a graph state $|G\rangle$, we can apply $X, Z$ operators at qubit $i$ by acting on qubits other than $i$.

We will use: $K_i |G\rangle = |G\rangle$

1. To apply $X_i$:

$$X_i |G\rangle = X_i K_i |G\rangle = \prod_{j \in N_G(i)} Z_j |G\rangle$$

where $N_G(i)$ are the neighbours of $i$ in the graph

2. To apply $Z_i$:

$$Z_i |G\rangle = Z_i K_{f(i)} |G\rangle = X_{f(i)} \prod_{j \in N_G(f(i)) \setminus i} Z_j |G\rangle$$

where $f(i)$ is the flow of $i$ and $N_G(f(i)) \setminus i$ are all the neighbours of $f(i)$ in the graph apart from $i$
How to apply an operator by acting on different qubits

- Can cancel a correction \textbf{after} measuring that qubit, provided \textbf{all} qubits \( j \) involved are still not measured!
- This \textbf{cannot} be done for the \( X \) operator (qubits both to the past and future according to the definition of flow).
- This \textbf{can} be done for the \( Z \) operator because of properties of the flow! (conditions 2 and 3)
Rules for adapting measurement angles

Measuring at $\theta$ qubit 1

Recall (previous lecture) the state before measurement is:

$$|\psi'\rangle_{12} = \wedge Z_{12} (|\psi\rangle_1 \otimes |+\rangle_2) = a |0+\rangle_{12} + b |1-\rangle_{12}$$

$$= |+\theta\rangle_1 X^{s_1} J(-\theta)_{2} |\psi\rangle_2 + |-\theta\rangle_1 X^{s_1} J(-\theta)_{2} |\psi\rangle_2$$
If we could have started with $Z_{s1}^1 |\psi\rangle_1$ state instead of $|\psi\rangle_1$:

$$
|\psi'\rangle_{12} = |+\theta\rangle_1 X_{s1} J(-\theta) Z_{s1}^2 |\psi\rangle_2 + |-\theta\rangle_1 X_{s1} J(-\theta) Z_{s1}^2 |\psi\rangle_2
$$

using that $J(-\theta) Z_{s1} = X_{s1} J(-\theta)$ and $X_{s1} X_{s1} = I$. Now, there is no random correction and any outcome of the measurement leads to the desired gate.

- Can view getting the “wrong” outcome $s_i = 1$ as a $Z$-correction on the initial state, and could cancel it by applying another $Z$ on that qubit.
- However, to do this we need to know $s_1$ which is the outcome of measuring qubit 1, and this (clearly) happens after the preparation of qubit 1.
Rules for adapting measurement angles

- Using stabilisers, we proved that we can apply $Z$ by acting on other qubits that are not yet measured!
- Instead of acting on the non-measured qubits, we can modify the measurement angles:

$$M_i^{\phi_i} X = M_i^{-\phi_i} \quad ; \quad M_i^{\phi_i} Z = M_i^{\phi_i + \pi}$$

since $X |+\phi_i\rangle = |+(-\phi_i)\rangle$ and $Z |+\phi_i\rangle = |-\phi_i\rangle = |+(\phi_i + \pi)\rangle$.

For qubit $i$ the corrections accumulate as follows:

1. An $X$-correction from $f^{-1}(i)$ the qubit that its flow is $i$.
2. A $Z$-correction from all qubits $j \neq i$ that their flow $f(j)$ is neighbour to $i$.

$$\phi'_i = (-1)^{s_{f^{-1}(i)}} \phi_i + \pi \left( \sum_{j : i \in N_G(f(j)) \mid j \neq i} s_j \right)$$
Adapting measurement angles
Example

- Each qubit has result $s_i$
- The flow is defined as $f(i) = i + 3$
- The order of measurements is the same as the labels
- We consider qubit 5:
  - $f^{-1}(5) = 2$
  - $N_G(5) = \{2, 4, 6, 8\}$
  - $X$-correction from $s_2$
  - $Z$-corrections from: $\{s_1, s_3\}$, since we look for $f^{-1}(\cdot)$ for each of the neighbours of 5. Qubit 2 has no past, qubit 8 has our qubit to its past, which leaves only the past of qubit 4 and qubit 6.

We then obtain the corrected measurement angle:

$$\phi'_5 = (-1)^{s_2} \phi_5 + \pi (s_1 + s_3)$$

**Note:** it depends only on outcomes of qubits measured before qubit 5.
**Brickwork State**

Universal cluster state that does not require $Z$ measurements for breaking the graph. Required for UBQC (see next lecture).

The basis for the global entangled state is the open graph $G_{n,m}$, where $m = 5 \mod 8$. This can be constructed with the following steps:
1. Assign to each vertex an index \((i, j)\), where \(1 \leq i \leq n\) is the row and \(1 \leq j \leq m\) is the column.

2. For each row \(i\) and for all \(1 \leq j \leq m - 1\) connect vertices \((i, j)\) and \((i, j + 1)\) with an edge.

3. For each column \(j = (3 \mod 8)\) and each odd row \(i\) connect vertices \((i, j)\) and \((i + 1, j)\) and also vertices \((i, j + 2)\) and \((i + 1, j + 2)\).

4. For each column \(j = (7 \mod 8)\) and each even row \(i\) connect vertices \((i, j)\) and \((i + 1, j)\) and also vertices \((i, j + 2)\) and \((i + 1, j + 2)\).

- We can merge different gates at this graph. An example of three unitaries of same size (sufficient for any universal gate) is the following:
- There order of measurements is column by column
- A universal gate set \( \{I, H, \pi/4, \wedge X\} \) can be built from same building block of 10 qubits

Implementation of a \( \pi/8 \) gate

Implementation of the identity

Implementation of a C\( \text{TRL} \)-\( X \)

Implementation of a Hadamard gate
Depth and Parallelism

- Storing quantum information (quantum states) without being affected from noise, is a very difficult task.
- It is therefore desirable to try make in parallel as many operations as possible.
- Also theoretically important aspect.
- Partial order of measurements means that any two qubits that are neither at the past nor the future of each other, can be measured in parallel.
- One can construct simplified graph that attempts to make as many measurements in parallel as possible.
- Depth of a flow $\Rightarrow$ length of longest chain w.r.t. POSET 
  [Chain: subset $S$ of set $C$ such that $S$ is a total order, i.e. $\forall e_i, e_j \in S$ either $e_i \leq e_j$ or $e_j \leq e_i$]
- Finding the flow with the smallest depth for a given computation $\Rightarrow$ maximum parallelise of MBQC pattern.
Start with a universal graph state such as the brickwork state. The computation is performed by measuring one-by-one the qubits using single-qubit bases either \( \{ |+\theta\rangle, |−\theta\rangle \} \) or \( \{ |0\rangle, |1\rangle \} \). Exact basis the qubits are measured depends on previous outcomes.

The order of measurements is determined by the flow \( f(i) \) along with partial order \( \preceq \).

Single qubit gates performed using \( J(\theta) \)-gate and along with \( \wedge Z \) is universal. Random corrections need to be cancelled.

A unitary \( U \) is implemented by a set of default angles \( \phi_i \) if all measurement had outcomes \( s_j = 0 \).

The actual basis that a qubit \( i \) is measured is modified, using the flow and the stabiliser properties we obtain:

\[
\phi'_i = (-1)^{s_{f^{-1}(i)}} \phi_i + \pi \left( \sum_{j : i \in N_G(f(j))} s_j \right)
\]
References for MBQC