Problem 1 - The CHSH inequality

In this problem you are asked to derive the CHSH inequality $S \leq 2$, where $S$ is the quantity defined in the lecture on entanglement and Bell’s inequality:

$$S = E_{00} - E_{01} + E_{10} + E_{11}$$

Here $E_{xy}$ is a correlator which was also defined in the lecture as $E_{xy} = \sum_{ab} ab P_{xy}(ab)$. In the proof you are to assume a local hidden variables model. This means that we are taking the correlator to be of the form $E_{xy} = \int A(x, \lambda) B(y, \lambda) \rho(\lambda) d\lambda$, where $\lambda$ is the hidden variable (or variables) and $\rho(\lambda)$ is a probability distribution over these hidden variables. Note that $A$ depends on $x$ but not on $y$ and viceversa for $B$. This is the idea behind locality. To aid in proving this we will break the problem into multiple subproblems:

A) Start by showing:

$$E_{00} - E_{01} = \int [A(x = 0, \lambda) B(y = 0, \lambda) (1 \pm A(x = 1, \lambda) B(y = 1, \lambda))] \rho(\lambda) d\lambda$$

$$- \int [A(x = 0, \lambda) B(y = 1, \lambda) (1 \pm A(x = 1, \lambda) B(y = 0, \lambda))] \rho(\lambda) d\lambda$$

B) Using the previous result, the triangle inequality\(^1\) and that $|A| \leq 1$, $|B| \leq 1$, prove:

$$|E_{00} - E_{01}| \leq \int (1 \pm A(x = 1, \lambda) B(y = 1, \lambda)) \rho(\lambda) d\lambda$$

$$+ \int (1 \pm A(x = 1, \lambda) B(y = 0, \lambda)) \rho(\lambda) d\lambda$$

C) From the previous inequality, and the fact that $\int \rho(\lambda) d\lambda = 1$, derive that:

$$|E_{00} - E_{01}| + |E_{10} + E_{11}| \leq 2$$

Now $S \leq 2$ is just a subcase of this, so we are done.

\(^1\)One form of the triangle inequality states that $|a - b| \leq a + b$. 
Problem 2 - Violating the CHSH inequality

Now that you have shown that $S \leq 2$, under the assumption of a local hidden variables model, we aim to show that given the assumptions of quantum mechanics and using a Bell state we are able to obtain $S = 2\sqrt{2}$.

As mentioned in the lecture, we are going to assume that Alice and Bob share the state $|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$.

A) Consider the operators:

$$
A_0 = Z, \quad A_1 = X
$$

$$
B_0 = \frac{1}{\sqrt{2}}(X + Z), \quad B_1 = \frac{1}{\sqrt{2}}(X - Z)
$$

Show that they are valid observables (i.e. that they are hermitian). If you think of the Bell state as an entanglement of 2 polarized photons, these observables correspond to different angles that we can measure for the polarization (polarization along the $X$ axis, the $Z$ axis, the axis in between $X$ and $Z$ etc).

B) Now we assume that when Alice’s measurement device is set to 0 she measures $A_0$ and when it is set to 1 she measures $A_1$. Similarly, Bob measures $B_0$ and $B_1$. Write the 4 possible observables, denoted $O_{00}$, $O_{01}$, $O_{10}$, $O_{11}$, of this setting.

C) Compute the expectation values of $O_{00}$, $O_{01}$, $O_{10}$, $O_{11}$ and check that:

$$
S = \langle O_{00} \rangle - \langle O_{01} \rangle + \langle O_{10} \rangle + \langle O_{11} \rangle = 2\sqrt{2}
$$

Remember that the expectation value of some observable $O$ with respect to a quantum state $|\psi\rangle$ is just $\langle \psi | O | \psi \rangle$. In our case the state is $|\Phi^+\rangle$ and we have 4 observables.

Problem 3

Given the states$^2$:

$$
|\Psi_1\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}
$$

$$
|\Psi_2\rangle = \frac{|001\rangle + |010\rangle + |100\rangle}{\sqrt{3}}
$$

Assume that for each state, the 3 qubits are shared among Alice, Bob and Eve. In both cases, trace out Eve’s qubit (the third one) from the states to compute the shared state that Alice and Bob have. In each case, would you say that this state is: separable, partially entangled or maximally entangled? (just give a qualitative answer)

$^2$State $|\Psi_1\rangle$ is known as the GHZ state, and $|\Psi_2\rangle$ is known as the W state.