Problem 1: (25 marks)

Given the state:

$$\Psi = \left( \frac{1}{2} + \epsilon \right)^{1/2} |00\rangle + \left( \frac{1}{2} - \epsilon \right)^{1/2} |11\rangle$$

for some $0 \leq \epsilon \leq 1/2$, notice that this is an approximate version of the maximally entangled state $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. We can quantify this closeness in two ways. One is to compute the overlap between this state and the $|\Phi^+\rangle$. The second, is to compute the $S$ quantity, from Tutorial 3, for the state.

**A)** Compute the “overlap” between $|\Psi\rangle$ and $|\Phi^+\rangle$, or, in other words, compute $\langle \Psi | \Phi^+ \rangle$. This quantity gives us an indication as to how close $|\Psi\rangle$ is to the maximally entangled state $|\Phi^+\rangle$, as a function of $\epsilon$.

**B)** Compute the $S$ quantity, from Tutorial 3, for $|\Psi\rangle$ and expand the expression into powers of $\epsilon$ up to leading order\(^1\). In other words, you can assume $\epsilon << 1/2$. Recall that the $S$ quantity was defined as:

$$S = E_{00} - E_{01} + E_{10} + E_{11}$$

Where $E_{xy}$ are the correlators, and we are using the observables from Tutorial 3:

$$A_0 = Z \quad \quad A_1 = X$$

$$B_0 = \frac{1}{\sqrt{2}}(X + Z) \quad \quad B_1 = \frac{1}{\sqrt{2}}(X - Z)$$

Problem 2: (25 marks)

Alice and Bob decide to play the following game: Alice flips a fair coin. If the coin lands on heads she will prepare, uniformly at random, one of the states $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$ and send it to Bob. If the coin lands

\(^1\)Recall that if $x << 1$, we have that $(1 + x)^\alpha \approx 1 + \alpha x$. 

on tails she will prepare, uniformly at random, one of the states $|\Phi^+\rangle$, $|\Phi^-\rangle$, $|\Psi^+\rangle$, $|\Psi^-\rangle$ and send it to Bob, where:

$$
|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \quad |\Phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}
$$

$$
|\Psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \quad |\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}
$$

Bob wins the game if he can determine what Alice’s coin landed on. Of course, he will try to infer this from the 2-qubit state that Alice sent him. What is the maximum probability that Bob wins this game? Justify your answer (Hint: try writing out the density matrices for the two situations).

**Problem 3: (25 marks)**

Many popular accounts of entanglement claim that it can be used in order to send information faster than the speed of light. Prove that this is not the case. Specifically, assume Alice and Bob share a general two-qubit state $|\Psi\rangle = \alpha_0 |00\rangle + \alpha_1 |01\rangle + \alpha_2 |10\rangle + \alpha_3 |11\rangle$, where $\sum_{i=0}^{3} |\alpha_i|^2 = 1$. The aim is to show that no matter what Alice does to her qubit (whether she does nothing, applies a unitary operation or measures it), the reduced state of Bob’s qubit remains the same (therefore, he cannot infer any information about Alice’s operations).

A) Assuming Alice does nothing to her qubit, compute the reduced density matrix of Bob’s qubit. Recall from the lectures that the reduced density matrix of Bob’s qubit is just $\text{Tr}_A(\rho)$, where $\text{Tr}_A$ is the partial trace over Alice’s qubit (tracing out Alice’s qubit) and $\rho$ is the state of the system. Start by first writing down $\rho$ explicitly.

B) Now assume Alice applies some general unitary $U$ to her qubit. Compute the reduced density matrix for Bob’s qubit. Compare with the previous result.

C) Lastly, assume Alice performs a measurement in the computational basis on her qubit. Compute again the reduced density matrix for Bob’s qubit. Compare with the previous results. How do you interpret these results? (Hint: If the state of the system prior to Alice’s measurement is $\rho$, then after measurement it will be $\rho' = (M_0 \otimes I)\rho(M_0 \otimes I) + (M_1 \otimes I)\rho(M_1 \otimes I)$, where $M_0 = |0\rangle \langle 0|$, $M_1 = |1\rangle \langle 1|$)

**Problem 4: (15 marks)**

A) Show that the following 2-qubit observables all commute with each other (pair-wise): $X \otimes I$, $I \otimes X$, $X \otimes X$. Next, show the same for: $X \otimes Z$, $Z \otimes X$, $(XZ) \otimes (ZX)$. Finally, prove the same thing for: $X \otimes X$, $Z \otimes Z$, $(XZ) \otimes (XZ)$. Recall that two observables, $A$ and $B$, commute if $[A,B] = AB - BA = 0$.

B) When 2 observables commute, they can be simultaneously measured. Mathematically, commuting observables share at least one common eigenvector. The common eigenvectors are the possible measurement outcomes, when measuring both observables. Consider $X \otimes X$ and $Z \otimes Z$, each having eigenvalues $+1$ and
−1 (so these are observables with 2 possible outcomes). These two observables will have a common +1 eigenvector, denoted $|\psi\rangle$, and a common −1 eigenvector, denoted $|\phi\rangle$. When they are measured simultaneously, on some two-qubit state, they will produce identical outcomes, collapsing the measured state to either $|\psi\rangle$, if both produced outcome +1 or to $|\phi\rangle$, if both produced outcome −1.

Compute $|\psi\rangle$ and $|\phi\rangle$. It is helpful to do so by expressing each state in the computational basis and then using the fact that these are eigenvectors. For instance, one can write $|\psi\rangle = \alpha_0 |00\rangle + \alpha_1 |01\rangle + \alpha_2 |10\rangle + \alpha_3 |11\rangle$, such that $\sum_{i=0}^{3} |\alpha_i|^2 = 1$ and $X \otimes X |\psi\rangle = |\psi\rangle$ and $Z \otimes Z |\psi\rangle = |\psi\rangle$. Use these facts to determine the $\alpha_i$ coefficients. Do the same for $|\phi\rangle$.

C) Suppose we have two commuting observables $A$ and $B$, each having eigenvalues +1 and −1. For simplicity, assume that either $A$ and $B$ have one common +1 eigenvector and one common −1 eigenvector (as in the previous example), or that $A$ has a +1 eigenvector which is a −1 eigenvector for $B$ and vice versa. Define the observable $C = AB$. Suppose we measure $C$ on some quantum state $|\psi\rangle$, obtaining outcome $c \in \{+1, -1\}$. Prove that had we instead measured first $A$ on $|\psi\rangle$, with outcome $a$ and then $B$ on the resulting state, with outcome $b$ then $c = ab$. In other words, show that measuring $C$ is the same as measuring $A$ and then $B$ and multiplying the outcomes.

**Problem 5: (10 marks)**

In the lectures, you were shown that one can use superdense coding in order to send 2 bits of information by only sending one qubit. You were also shown that quantum teleportation involves “sending” (teleporting) one qubit of quantum information by transmitting 2 classical bits. Suppose there exists a procedure, which we will call hyperdense coding, with which one can send $k$ bits of information by sending a single qubit. Show, either for $k = 3$ or for $k = 4$, that if such a procedure existed, one could send an unbounded amount of information by sending a single qubit\(^2\). In other words, for any $N > 0$, show that there exists a procedure through which Alice can send $N$ bits of information to Bob by only sending him one qubit (and nothing else, not even classical bits).

**Total marks: 100**

**BONUS – Problem 6: (15 marks)**

Alice and Bob decide to play yet another game. Suppose we have a 3×3 grid, as in Subfigure 1a, which we shall refer to as a *special square*. Alice and Bob are separated so that they cannot communicate. Alice will be given a number $i \in \{1, 2, 3\}$, representing one of the 3 lines in the square, whereas Bob is given a number $j \in \{1, 2, 3\}$, representing one of the 3 columns in the square. Alice must then provide a 3-tuple $(a_1, a_2, a_3)$, where each $a_i$ is either + or − and the number of + symbols must be even, whereas the number of − symbols must be odd. For instance, a valid response for Alice is $(-, +, +)$. If she received line $i = 1$, then line 1 of the square is completed with $(-, +, +)$.

\(^2\)You do not need to prove this for both $k = 3$ and $k = 4$, either one is sufficient for full marks.
Conversely, Bob must provide a 3-tuple \((b_1, b_2, b_3)\), also containing + and − symbols, but such that the number of + symbols is \textit{odd} and the number of − symbols is \textit{even}. For instance, a valid response for Bob is \((+, +, +)\). If he received line \(j = 3\), then line 3 of the square is completed with \((+, +, +)\).

Alice and Bob win the game if they put the same symbol in their common square. In other words \(a_j = b_i\).

The example described above is shown in Subfigure 1b, where both Alice and Bob put a + in the upper right corner of the square. Alice’s responses are in green, Bob’s are in red, whereas their common square is purple. However, as shown in Subfigure 1c, if Alice provided the same tuple \((-+, +, +)\), but Bob provided \((-+, +, -)\) then they would lose, since \(a_3 = +, \text{ but } b_1 = -\).

\[
\begin{bmatrix}
X \otimes I & -X \otimes X & I \otimes X \\
X \otimes Z & -(XZ) \otimes (ZX) & Z \otimes X \\
I \otimes Z & -Z \otimes Z & Z \otimes I
\end{bmatrix}
\]

Notice, from problem 4, that the observables in each row (column) commute pair-wise. This means that all observables in a row (column) can be measured simultaneously. Alice will simultaneously measure the observables from row \(i\) on her two qubits, whereas Bob will simultaneously measure the observables from row \(j\) on his two qubits. An outcome of +1, for a measurement, is taken as the symbol +, whereas an outcome of −1 is taken as the symbol −.
Using problem 4C, show that Alice’s outcomes always produce an even number of + symbols and that Bob’s outcomes always produce an odd number of + symbols. This ensures that both Alice and Bob provide valid tuples.

Finally, show that, because they are sharing Bell states, Alice and Bob will always produce the same outcome on their common square. In other words, show that when they both measure the same observables on their respective qubits, they will obtain identical results.