Introduction to Quantum Computing

Tutorial 1

2nd October 2017

Problem 1

Given the matrices:

\[
X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
\]

A) Show that they are hermitian and unitary.

B) Compute the eigenvalues and eigenvectors of these matrices.

Problem 2

A) Express the states \(|+\rangle = H |0\rangle\) and \(|-\rangle = H |1\rangle\) in the computational basis. (where \(H\) is the matrix from the previous problem)

B) Show that \((|+\rangle, |-\rangle)\) is an orthonormal basis for a 2-dimensional Hilbert space.

C) Express the 2-qubit state \(|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}\) in the \((|++\rangle, |+-\rangle, |--\rangle, |--\rangle)\) basis.

Problem 3

The trace of an operator \(A\), acting on some vector space \(V\), can be written as \(\text{Tr}(A) = \sum_{i \in B} \langle i | A | i \rangle\), where \(B\) is an orthonormal basis of \(V\). When we write \(\text{Tr}(A) = \sum_i A_{ii}\) we are considering the canonical basis.

Show that the trace function is independent of the choice of the basis. Concretely, show that for an arbitrary orthonormal basis \(B'\), \(\text{Tr}(A) = \sum_{j \in B'} \langle j | A | j \rangle\). (Hint: Completeness relation)

1. The \(X\), \(Y\), \(Z\) matrices are known as Pauli matrices and \(H\) is known as the Hadamard matrix.
2. The state \(|\Phi^+\rangle\) is known as a Bell state.
Problem 4

A) Given the operators $M_0 = |0⟩⟨0|$, $M_1 = |0⟩⟨1|$, do they form a valid set of measurement operators? Why/Why not? If so, does the set constitute a projective measurement? Why/Why not?

B) What about $M_0 = |0⟩⟨+|$, $M_1 = |1⟩⟨−|$?

C) What about $M_0 = |0⟩⟨0|$, $M_1 = |+⟩⟨+|$?

Problem 5*

For a mixed state given by the density matrix $ρ$, show that $\text{Tr}(ρ^2) ≤ 1$, with equality when $ρ$ is a pure state. (Hint: You can solve this either by considering $ρ$ as a statistical mixture of some ensemble of states $\{p_i, |ψ_i⟩\}$ so $ρ = \sum_i p_i |ψ_i⟩⟨ψ_i|$, or simply using the properties of the density matrix, i.e. that it is hermitian, positive semi-definite and that $\text{Tr}(ρ) = 1$)